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Non-topological non-commutativity in string theory

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Quantization of coordinates leads to the non-commutative product of deformation quantization, but is also at the roots of string theory, for which space-time coordinates become the dynamical fields of a two-dimensional conformal quantum field theory. Appositely, open string diagrams provided the inspiration for Kontsevich's solution of the long-standing problem of quantization of Poisson geometry by virtue of his formality theorem. In the context of D-brane physics non-commutativity is not limited, however, to the topolocial sector. We show that non-commutative effective actions still make sense when associativity is lost and establish a generalized Connes-Flato-Sternheimer condition through second order in a derivative expansion. The measure in general curved backgrounds is naturally provided by the Born–Infeld action and reduces to the symplectic measure in the topological limit, but remains non-singular even for degenerate Poisson structures. Analogous superspace deformations by RR–fields are also discussed.

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1 Introduction

The non-commutative product of deformation quantization [1, 2] can be derived from string theory in a topological limit where the space-time metric is small as compared to the anti-symmetric B-field (the ancestor of the Poisson bi-vector) [3–5]. The non-commutative product thus amounts to a summation of the leading B-field contributions to the effective action. In the non-symplectic case this interpretation is spoiled, however, by the absence of a canonical measure. From the string theory point of view, on the other hand, associativity is lost for generic backgrounds [6], but the Born-Infeld action provides a canonical measure [4, 7]. We show that the concept of effective actions does not require associativity, but rather a generalized Connes–Flato–Sternheimer condition called cyclicity [8, 9], i.e. commutativity and associativity up to surface terms [10]. Cyclicity implies, however, a compatibility condition between the star product and the measure [9], which for Born-Infeld turns out to be equivalent to the generalized Maxwell equation for the gauge field on the D-brane [7, 10]. In [10] we found that cyclicity also requires a gauge modification of the Kontsevich product at second derivative order in a derivative expansion and we discussed the D-brane physics related to these mathematical structures.

In section 2 of this note we review some aspects of deformation quantization and formality in simple terms by illustrating the emergence of Hochschild cocycles, Gerstenhaber brackets and gauge transformations accompanying diffeomorphisms in derviative expansions. In section 3 we discuss the stringy origin

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of these structures and their interpretation in terms of effective actions, which requires the existence of a measure and a generalized Connes–Flato–Sternheimer property. While associativity is restricted to Poisson geometry, string theory naturally introduces the Born–Infeld measure and keeps cyclicity, at least through second derivative order, independently of associativity and without a topological limit. We observe that the results found in [10] straightforwardly extend to non-constant dilaton backgrounds. In section 5 we discuss the Berkovits string in general RR backgrounds and the resulting deformation of superspace. In section 6 we conclude with a discussion of open problems and work to be done.

2 Deformation quantization, Kontsevich product and formality

The idea of deformation quantization is to emulate the operator product of quantum mechanics by an associative product $f\star g$ of phase space functions $f,g\in C^\infty(M)$ with

$$f\star g=f\,g+\frac{i}{2}\hbar\{f,g\}_{PB}+\mathcal{O}(\hbar^2)\quad\Rightarrow\quad\lim_{\hbar\to 0}\frac{f\star g-g\star f}{i\hbar}=\{f,g\}_{PB},\tag{2.1}$$

where the Poisson bracket can be written for arbitrary phase space coordinates x^μ as a bi-derivation $\{f,g\}_{PB}=\Theta^{\mu\nu}(x)\partial_\mu f\partial_\nu g$ in terms of a bi-vector field $\Theta\in\Lambda^2TM$.

2.1 Polyvectors and the Schouten–Nijenhuis bracket

Elements $X\in\Lambda^\bullet TM$ of the exterior algebra over the tangent space TM are called polyvector fields and there is a bilinear operation, the Schouthen–Nijenhuis (SN) bracket

$$[X^{(p)},Y^{(q)}]\in\Lambda^{p+q-1}TM\quad\text{for}\quad X^{(p)}\in\Lambda^pTM\quad\text{and}\quad Y^{(q)}\in\Lambda^qTM,\tag{2.2}$$

that extends the Lie bracket of vector fields to a graded bi-derivation of degree -1 on $T\in\Lambda^\bullet TM$. The Jacobi identity of the Poisson bracket is equivalent to the vanishing of the SN bracket $[\Theta,\Theta]$,

$$\circlearrowright\!\!\!\!\!\sum\{\{f,g\}_{PB},h\}_{PB}=0\quad\Leftrightarrow\quad[\Theta,\Theta]=0\quad\text{with}\quad[\Theta,\Theta]^{\mu\alpha\beta}=\frac{2}{3}\circlearrowright\!\!\!\!\!\sum\Theta^{\mu\rho}\partial_\rho\Theta^{\alpha\beta}.\tag{2.3}$$

Lie derivatives in the direction of $\xi\in TM$ can also be written in terms of the SN bracket $\mathcal{L}_\xi X=[\xi,X]$ for all polyvector fields $X\in\Lambda^\bullet TM$.

2.2 Moyal product and Kontsevich graphs

In case of constant Θ, and hence in particular locally for Darboux coordinates, deformation quantization can be achieved by the Moyal product

$$(f\star g)(x)=\exp\left(\tfrac{i}{2}\hbar\,\Theta^{\mu\nu}\partial_{y^\mu}\partial_{z^\nu}\right)f(y)g(z)\,\Big|_{y\,=\,z\,=\,x}\tag{2.4}$$

After a general change of coordinates in phase space Θ will not stay constant, which motivates the consideration of deformation quantization for general Θ. For the symplectic case $\det\Theta\neq 0$ the existence of a star product has been shown by De Wilde and Lecompte [11] and the first construction is due to Fedosov [12]. Some details and a historical assessment with references can be found in the review [2]. For the case of a general Poisson structure Θ, which by definition obeys $[\Theta,\Theta]=0$, the construction of an associative product is due to Kontsevich [1] and will now be discussed in more detail. Associativity of this product is, in fact, a corollary of the formality theorem, which establishes a quasi-isomorphism of L_∞ algebras. The formality map U maps polyvector fields T_i to polydifferential operators $U(T_1,\ldots,T_n)=\sum_\Gamma w_\Gamma D_\Gamma$ and is constructed in terms of graphs Γ and coefficients w_Γ. The coefficients w_Γ are defined by convergent integrals inspired by open string Feynman diagrams (cf. section 3) with functions inserted on the real line and polyvector fields in the upper half plane as illustrated in fig. 2.1, where the derivatives of the bidifferential operators correspond to the arrows pointing at f and g. The first two graphs fig. 2.1a give the order Θ and Θ^2 terms of the Moyal product, while fig. 2.1b yields first derivative corrections for non-constant Θ. The latter will be worked out explicitly below. The precise relation of Kontsevich's construction to correlation functions of topological sigma models is due to Cattaneo and Felder [5].

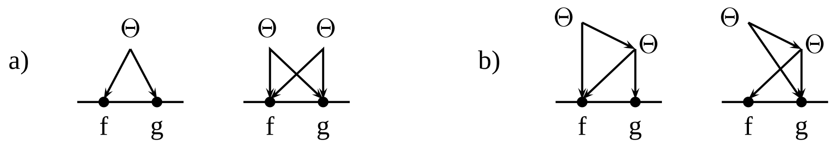

Fig. 2.1 Kontsevich graphs for a) Moyal-type contributions and b) derivative corrections, respectively.

2.3 Hochschild cohomology, Gerstenhaber bracket, and the formality theorem

Rather than giving abstract definitions of the involved mathematical structures we now illustrate how they automatically show up in simple calculations. We ignore for a moment the relation (2.1) to Poisson brackets and consider a general deformation of the product

$$f \star g = fg + \hbar B_1(f,g) + \mathcal{O}(\hbar^2) \qquad \text{with} \qquad B_1(f,g) = B^{\mu\nu} f_\mu g_\nu, \quad f_\mu \equiv \partial_{x^\mu} f, \tag{2.5}$$

where derivatives of functions are abbreviated by subscripts. The $\mathcal{O}(\hbar)$ contribution to the associator,

$$f \star (g \star h) - (f \star g) \star h = \hbar \Big(f B_1(g,h) - B_1(fg,h) + B_1(f,gh) - B_1(f,g)h \Big) + \mathcal{O}(\hbar^2), \tag{2.6}$$

has exactly the form of a Hochschild coboundary [1]

$$(\delta C)(f_0, \ldots, f_p) = f_0 C(f_1, \ldots, f_p) - C(f_0 f_1, \ldots, f_p) + C(f_0, f_1 f_2, \ldots, f_p) - \ldots \tag{2.7}$$

There are, however, equivalences of the resulting deformed associative algebras due to invertible maps $f \to Df$ with differential operators

$$D = 1 + \hbar(D_1^\mu \partial_\mu + D_1^{\mu\nu} \partial_\mu \partial_\nu + \ldots) + \hbar^2 (D_2^\mu \partial_\mu + \ldots) + \ldots \tag{2.8}$$

that respect the unit element $D1 = 1$. They lead to the following modification of the star product,

$$f \to Df \qquad \Rightarrow \qquad f \star' g = D(D^{-1} f \star D^{-1} g) \tag{2.9}$$

and hence $B_1'(f,g) - B_1(f,g) = -fD_1(g) + D_1(fg) - D_1(f)g$, at order $\hbar$, which is again a Hochschild coboundary. For the special case $D_1 = D_1^{\mu\nu} \partial_\mu \partial_\nu$ this implies the gauge equivalence $B_1'(f,g) - B_1(f,g) = D_1^{\mu\nu} f_\mu g_\nu$ so that for the first order bidifferential operator $B_1(f,g) = B_1^{\mu\nu} f_\mu g_\nu$ of eq. (2.5) the symmetric part of $B_1^{\mu\nu}$ can be gauged away with $D_1^{\mu\nu} = B_1^{(\mu\nu)}$. With the choice $B_1^{\mu\nu} = \frac{i}{2} \Theta^{\mu\nu}$ we thus recover (2.1).

Returning to the Kontsevich graphs fig. 2.1 we now want to work out the derivative corrections that are needed for associativity at order $\hbar^2$. For this purpose we define the Moyal part

$$[f \star g] \equiv fg + i\tfrac{\hbar}{2} \Theta^{\mu\nu} f_\mu g_\nu - \tfrac{\hbar^2}{8} \Theta^{\mu\alpha} \Theta^{\nu\beta} f_{\mu\alpha} g_{\nu\beta} - \ldots \tag{2.10}$$

of a product $f \star g$ as the result of dropping all terms with derivatives acting on Θ. Then

$$f \star g = [f \star g] - \hbar^2\, \Theta^{\mu\rho} \partial_\rho \Theta^{\alpha\beta} (a\, [f_{\alpha\mu} \star g_\beta] + b\, [f_\alpha \star g_{\beta\mu}]) + \mathcal{O}\big(\partial^2\big) \tag{2.11}$$

where $\mathcal{O}\big(\partial^2\big)$ only counts derivatives acting on Θ and the coefficients ω_Γ of the two graphs in fig. 2.1b are a and b, respectively. Instead of determining these coefficients from integrals over Θ in the upper half plane we determine them by imposing associativity. The first derivative order part of $f \star (g \star h)$ is

$$-\hbar^2 X^{\mu\alpha\beta} \Big[a f_{\mu\alpha} \star (g \star h)_\beta + b f_\alpha \star (g \star h)_{\mu\beta} + \frac{1}{4} f_\mu \star (g_\alpha \star h_\beta) + a f \star g_{\mu\alpha} \star h_\beta + b f \star g_\alpha \star h_{\mu\beta} \Big] \tag{2.12}$$

with $X^{\mu\alpha\beta} = \Theta^{\mu\rho} \partial_\rho \Theta^{\alpha\beta}$, and the $\mathcal{O}(\partial)$ contributions to $(f \star g) \star h$ are

$$-\hbar^2 X^{\mu\alpha\beta} \big[a (f \star g)_{\mu\alpha} \star h_\beta + b\, (f \star g)_\alpha \star h_{\mu\beta} - \frac{1}{4} (f_\alpha \star g_\beta) \star h_\mu + a f_{\mu\alpha} \star g_\beta \star h + b f_\alpha \star g_{\mu\beta} \star h \big] \tag{2.13}$$

so that $f \star (g \star h) - (f \star g) \star h = \hbar^2 [f_\mu * g_\alpha * h_\beta] \Big((a - \frac{1}{4}) X^{\mu\alpha\beta} + (a - b) X^{\alpha\mu\beta} - (b + \frac{1}{4}) X^{\beta\mu\alpha} \Big)$. Associativity implies that the coefficient of $[f_\mu * g_\alpha * h_\beta]$ vanishes. Using the antisymmetry of Θ we first observe that X

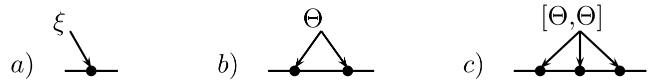

Fig. 2.2 Dressings of a) Lie derivative, b) Poisson bracket and c) associator, respectively.

cannot be totally antisymmetric. Thus symmetrization in $\mu\alpha$, $\alpha\beta$ and $\beta\mu$ implies $b = 2a - \frac{1}{4},\ a = 2b + \frac{1}{4}$ and $a + b = 0$, respectively. The unique solution is $a = -b = \frac{1}{12}$. Hence

$$f \star (g \star h) - (f \star g) \star h = -\frac{1}{6}\hbar^2 [f_\mu * g_\alpha * h_\beta] \sum_{\mu\alpha\beta} \Theta^{\mu\rho}\partial_\rho \Theta^{\alpha\beta} + \mathcal{O}(\partial^2) \tag{2.14}$$

so that $[\Theta,\Theta] = 0$ is a necessary condition for the existence of an associative deformation. The Kontsevich product through second derivative order (setting $\hbar = 1$)

$$\begin{aligned} f \star g \ &= \ [f \star g] - \frac{1}{12}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\rho}\big[f_{\mu\nu}\star g_\rho + f_\rho \star g_{\mu\nu}\big] + \frac{1}{24}\partial_\sigma\Theta^{\mu\rho}\partial_\rho\Theta^{\nu\sigma}[f_\mu \star g_\nu] \\ &\quad + \frac{\mathrm{i}}{48}\Theta^{\mu\gamma}\partial_\gamma\Theta^{\nu\delta}\partial_\delta\Theta^{\rho\lambda}\ [\,f_{\mu\rho}\star g_{\nu\lambda} - f_{\nu\lambda}\star g_{\mu\rho}] - \frac{\mathrm{i}}{48}\Theta^{\mu\gamma}\Theta^{\nu\delta}\partial_\gamma\partial_\delta\Theta^{\rho\lambda}\ \big[f_{\mu\nu\rho}\star g_\lambda - f_\lambda \star g_{\mu\nu\rho}\big] \\ &\quad + \frac{1}{2}\frac{1}{12^2}(\Theta^{\mu\gamma}\partial_\gamma\Theta^{\rho\lambda})(\Theta^{\nu\delta}\partial_\delta\Theta^{\sigma\tau})\big[f_{\mu\rho\nu\sigma}\star g_{\lambda\tau} + 2f_{\mu\rho\tau}\star g_{\lambda\nu\sigma} + f_{\lambda\tau}\star g_{\mu\rho\nu\sigma}\big] + \mathcal{O}(\partial^3) \end{aligned} \tag{2.15}$$

has been determined in [10] using the known coefficient $\frac{1}{24}$ of the gauge term and symmetry under complex conjugation combined with the exchange of f and g. Note that each term in (2.15) comprises the contributions of an infinite number of graphs with Moyal-type additions to the classical part of $\star$, since this formula holds to all orders in the undifferentiated Θ's.

The Gerstenhaber bracket of polydifferential operators P_i is the commutator $[P_1, P_2]$ with respect to an appropriate definition of the composition $P_1 \circ P_2$ of degree -1. For bidifferential operators the bracket thus yields a tridifferential operator, and in the special case $P_1 = P_2 = \star$ the bracket becomes proportional to the associator $\frac{1}{2}[\star,\star](f,g,h) = f\star(g\star h) - (f\star g)\star h$. The formality map can be regarded as a dressing, or quantization, of polyderivations $T_i \in \Lambda^\bullet TM$ to higher order polydifferential operators P_i. The formality theorem ensures that this map is an $\mathcal{L}_\infty$ quasi-isomorphism where the (homotopy) Lie algebra structures are related to the SN bracket and the Gerstenhaber bracket, respectively.

The cases of vector fields ξ, Poisson tensors Θ and rank three tensors $J \in \Lambda^3 TM$ shown in fig. 2.2 are of particular interest. The quantization of Θ yields the star product (2.15). Since the SN bracket is mapped to the Gerstenhaber bracket, $J = [\Theta,\Theta]$ as well as its quantization vanish in the case of a Poisson structure $[\Theta,\Theta] = 0$. Since $[\star,\star]$ is the associator this establishes associativity of the Kontsevich product (compare fig. 2.2c to our result (2.14) at leading order $\hbar^2$).

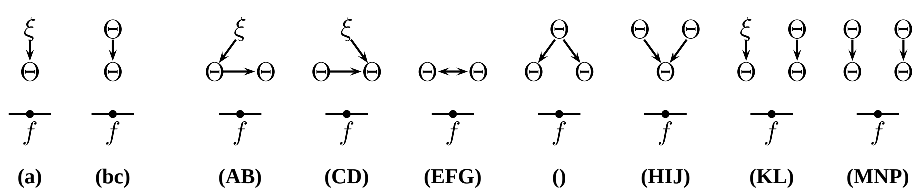

Fig. 2.3 Sample graphs for dressed coordinate transformations through second derivative order.

For vector fields ξ the classical term is the Lie derivative, which amounts to a change of coordinates. Its quantization yields an equivalence transformation D_ξ of the form (2.9) so that the Kontsevich product transforms covariantly under changes of coordinates only up to gauge equivalence. For infinitesimal transformations

$$\partial_t(f \star_t g) = D_\xi f \star g + f \star D_\xi g - D_\xi(f \star g) \qquad \text{with} \qquad \dot{x}^\mu = \xi^\mu. \tag{2.16}$$

In fig. 2.3 we enumerate the graphs that contribute to infinitesimal transformations (i.e. linear in ξ) through second derivative order in Θ. Note that the additional lines corresponding to derivatives acting on ξ and f

can lead to different tensor structure, as indicated by coefficients $A, \ldots, P$ for the terms with two derivatives on Θ's, plus an infinite number of additional Moyal-type contributions. Through first derivative order

$$D_\xi = \xi^\alpha \partial_\alpha + \frac{1}{24} \xi^\alpha_{\mu\rho} \Theta^{\mu\nu} \partial_\nu \Theta^{\rho\beta} \partial_\alpha \partial_\beta + \mathcal{O}(\partial^2). \tag{2.17}$$

At second derivative order the graphs define differential operators D_ξ containing (non-Moyal) terms with up to five derivatives, $D_\xi f = \xi^\alpha f_\alpha + \ldots + P \xi^\alpha_{\rho\sigma} \partial_\nu \Theta^{\beta\mu} \partial_\mu \Theta^{\rho\gamma} \Theta^{\delta\nu} \partial_\nu \Theta^{\sigma\varepsilon} f_{\alpha\beta\gamma\delta\varepsilon}$ but many coefficients may be zero.

3 Open strings, Born–Infeld electrodynamics and non-commutativity

In order to relate the Kontsevich product (2.15) to string theory we start with the Polyakov action for closed strings moving in a curved background with 2-form field B. In conformal gauge

$$S_P = \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \, \partial X^\mu \bar{\partial} X^\nu \Big(g_{\mu\nu}(X) + B_{\mu\nu}(X) \Big), \tag{3.1}$$

where $X^\mu : \Sigma \longrightarrow M$ maps the closed world sheet Σ to the target manifold M. Note that this action is invariant under the gauge transformation $\delta_\Lambda B = d\Lambda$.

When we consider open strings, we have to introduce world sheets with boundaries and specify a hypersurface in M, i.e. a D-brane, to which the end points of open strings are mapped. In the following we will only consider space-filling branes. By Stokes' theorem, (3.1) is not gauge invariant anymore, $\int_\Sigma X^* \delta_\Lambda B = \int_{\partial\Sigma} X^* \Lambda$, and we have to introduce a compensator field A at the boundary, which turns out to be a $U(1)$ gauge field with field strength $F = dA$. The associated action,

$$S_A = \int_{\partial\Sigma} X^* A = \int_{\partial\Sigma} dt \, \partial_t X^\mu A_\mu(X) = \int_\Sigma X^* F, \tag{3.2}$$

then restores gauge invariance of (3.1) by setting $\delta_{\Lambda,\lambda} A = -\frac{1}{2\pi\alpha'} \Lambda + d\lambda$.

As a consequence of gauge symmetry the effective action depends on the fields A and B only through the gauge invariant quantities $\mathcal{F} = B + 2\pi\alpha' F$ and $H = dB = d\mathcal{F}$. For slowly varying fields $\mathcal{F}$ and g the effective theory on the D-brane is Born–Infeld electrodynamics [13] governed by

$$S_{BI} = \int_M d^D x \, \sqrt{\det(g_{\mu\nu} + \mathcal{F}_{\mu\nu})} \, . \tag{3.3}$$

Let us have a closer look at the quantization of (3.1) and (3.2) on the upper half plane, conformally equivalent to the disk. We split the embedding map into fluctuations around a constant mode, $X^\mu(z,\bar{z}) = x^\mu + \zeta^\mu(z,\bar{z})$, and organize the perturbative quantization in terms of a derivative expansion in the background fields. Moreover, we regard the metric $g(x)$ and the curvature $\mathcal{F}(x)$ as a classical background in order to ensure conformal invariance.

The variation of the action requires the mixed Dirichlet–Neumann boundary condition

$$g_{\mu\nu} \partial_t X^\nu - \mathcal{F}_{\mu\nu} \partial_n X^\nu \big|_{\partial\Sigma} = 0 \, , \tag{3.4}$$

which leads to the following propagator for fluctuations at the boundary ($\tau, \tau' \in \partial\Sigma$):

$$\langle \zeta^\mu(\tau) \, \zeta^\nu(\tau') \rangle = -\frac{1}{2\pi} \Big\{ G^{\mu\nu}(x) \ln |\tau - \tau'|^2 + i\pi \Theta^{\mu\nu}(x) \, \epsilon(\tau - \tau') \Big\}, \tag{3.5}$$

where we introduced $G^{(\mu\nu)} + \Theta^{[\mu\nu]} := (g_{\mu\nu} + \mathcal{F}_{\mu\nu})^{-1}$ and the sign function $\epsilon(\tau) = \tau/|\tau|$.

In the limit when $g_{\mu\nu}$ vanishes (with $\mathcal{F}_{\mu\nu}$ kept finite) [4], the action $S_P + S_A$ is topological. Only the second part in (3.5) survives, and the non-commutative product on the D-brane world volume becomes apparent. For constant backgrounds it is the Moyal product. For varying backgrounds we notice, however, that the Einstein equations for the background fields require $H = d\mathcal{F} = 0$ in the topological limit [14], i.e. $\mathcal{F}$ is a symplectic form with Poisson structure $\Theta = \mathcal{F}^{-1}$. The resulting non-commutative product is then the associative product (2.15) due to Kontsevich [1].

4 Associativity, cyclic invariance and effective actions

From the string theory point of view, the assumption of $\Theta(x)$ being a Poisson structure is not natural. The only condition on the background fields should come from conformal invariance, or equivalently the classical equations of motion. Therefore, it is preferable to define the non-commutative product without taking the topological limit. It is clear that the first term in the propagator (3.5) should play a secondary rôle in this definition, which suggests to consider two (off-shell) vertex operators at a distance $\tau' - \tau = 1$ [7], i.e.

$$f(x) \circ g(x) := \frac{1}{\sqrt{|g+\mathcal{F}|}} \int \mathcal{D}\zeta \; e^{-S[X=x+\zeta]} \, f(X(0)) \, g(X(1)). \tag{4.1}$$

The Born–Infeld measure in the prefactor is cancelled by (world sheet) 1-loop diagrams. At higher derivative orders of the background fields the measure gets corrected [15].

Let us comment on some properties of this product.

- An immediate consequence of giving up on $\Theta(x)$ being Poisson is the *loss of associativity*, so that a sum over different configurations of brackets will appear in open string scattering amplitudes and in the effective action. In the topological limit, the non-commutative product (4.1) becomes the Kontsevich product, up to gauge equivalence, $D(f \star g) = Df \circ Dg$, so that associativity is restored.
- As was argued in [10], the variational principle for the low-energy effective theory requires that the non-commutative product is *cyclic*, i.e.

  $$\int_M \Omega \; f \circ g = \int_M \Omega \; f \cdot g \quad \text{and} \quad \int_M \Omega \; (f \circ g) \circ h = \int_M \Omega \; f \circ (g \circ h), \tag{4.2}$$

  where Ω is a measure, which requires $\partial_\mu(\Omega\, \Theta^{\mu\nu}) = 0$. From a string theory point of view the measure Ω is the Born–Infeld measure that appeared in (3.3), i.e. $\Omega = \sqrt{|g+\mathcal{F}|}$, and cyclicity follows from the generalized Maxwell equation associated with the Born–Infeld action (3.3):

  $$\partial_\mu(\sqrt{|g+\mathcal{F}|}\Theta^{\mu\nu}) = 0 \qquad \Longleftrightarrow \qquad G^{\rho\sigma} D_\rho \mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma} H_{\rho\sigma}{}^{\lambda} \mathcal{F}_{\lambda\mu} = 0. \tag{4.3}$$

  This is in line with the assumption of a classical background, which ensures conformal invariance and, in particular, cyclic invariance of disk amplitudes. Notice that if we include the dilaton ϕ in the background, the measure is modified to $e^{-\phi}\sqrt{|g+\mathcal{F}|}$.

  For Poisson structures the second condition in (4.2) follows from associativity, and the first is due to Connes–Flato–Sternheimer [16]. In fact, for any volume form Ω subject to $\partial_\mu(\Omega\Theta^{\mu\nu}) = 0$ there exists a star-product that satisfies cyclic invariance (4.2) [9]. However, in contrast to the physical context above, there is no canonical measure for Poisson structures.
- In [7] an explicit computation of the product (4.1) was given to first derivative order, $\partial\Theta$, in the background field, but to all orders in Θ. In [10] it was shown that the cyclic invariance (4.2) uniquely fixes the non-commutative product to second derivative order, with the result

  $$f \circ g = f \star g - \frac{1}{24}\Theta^{\mu\rho}\,\Theta^{\nu\sigma}\,\partial_\rho\partial_\sigma(\log\Omega)\, f_\mu\, g_\nu. \tag{4.4}$$

  The first contribution is the same expression (2.15) as the Kontsevich product but without the Poisson constraint on Θ and the second is a gauge term that is needed to ensure cyclic invariance.

If we want to use the non-commutative product (4.1) to compute string S-matrix elements we have to impose on-shell conditions not only on the background fields but also on the vertex operator insertions. In the present context the vertex operators are functions, $f(X)$, and thus the on-shell condition is the one for an open string tachyon $T(x)$

$$\Box T = \frac{1}{\Omega}\,\partial_\mu\big(\Omega\, G^{\mu\nu}\partial_\nu T\big) = -\frac{1}{\alpha'}\,T. \tag{4.5}$$

This fixes the kinetic term for the low-energy effective action. The result is

$$S = -\frac{1}{2g_o^2}\int_M \Omega\,\Big\{G^{\mu\nu}\,\partial_\mu T\,\partial_\nu T - \frac{1}{\alpha'}\,T^2 - \sqrt{\frac{8}{9\alpha'}}\,T\,(T\circ T)\Big\}, \tag{4.6}$$

where the cubic tachyon interaction was found in [17] by computing 3-point amplitudes.

5 Superstrings and non-anticommutative superspace

The superstring in Green-Schwarz (GS) related formulations is an embedding of a string in superspace. It thus appears natural that, in addition to non-commutativity of space-time coordinates, there should be a mechanism that deforms the anticommutation of the fermionic superspace coordinates. Indeed such a mechanism exists. Independently of string theory, special cases of non-anticommuting supercoordinates were already considered by van Nieuwenhuizen and others in [18] ($N = \frac{1}{2}$ SUSY, see [19]). A more general ansatz was presented in [20]. After indications in [21, 22] that similar structures originate from the superstring, this could eventually be shown in [23] for a string in four dimensions (with six dimensions compactified on a Calabi-Yau) and was generalized in [24] to ten dimensions. In both cases a constant RR-field-strength was considered and turned out to be responsible for the nonanticommutativity of the supercoordinates. The calculations where performed in different versions of the covariant superstring [25–27]. This non-(anti)commutativity can again be implemented via a star product, now on superspace (see [28] and references therein). For non-constant background fields (but in the topological limit), this corresponds to a graded generalization of Kontsevich's associative star product. A derivation from a σ-model with super-targetspace along the lines of Cattaneo and Felder [5] was presented in [29]. The effect of a constant RR-potential (not field strength) on the deformation of the bosonic space was already studied in [31]. In the following we sketch how non-anticommutativity of superspace arises from the Berkovits pure spinor superstring [24].

Although we will consider an open string with type I supersymmetry, we want to couple it to the type II bulk fields (see e.g. [31]). In particular the RR-fields belong to the bulk and will take over the role of the B-field in the fermionic case. It is therefore necessary to embed the string into a type II superspace with coordinates $x^M = (x^m, \theta^\mu, \hat{\theta}^{\hat{\mu}})$. In this section Greek letters will be reserved for fermionic indices while bosonic indices are denoted by Latin letters. In conformal gauge, the GS action in flat background reads

$$\begin{aligned} S_{GS} &\equiv \int \mathrm{d}^2\mathrm{z} \quad \tfrac{1}{2}\Pi^{\mathrm{a}}_{\mathrm{z}}\eta_{\mathrm{ab}}\Pi^{\mathrm{b}}_{\bar{\mathrm{z}}} + \mathcal{L}_{\mathrm{WZ}} \qquad \text{(conformal gauge)} \\ \mathcal{L}_{WZ} &\equiv -\tfrac{1}{2}\Pi^a_z\left(\theta\gamma_a\bar{\partial}\theta - \hat{\theta}\gamma_a\bar{\partial}\hat{\theta}\right) + \tfrac{1}{2}(\theta\gamma^a\partial\theta)(\hat{\theta}\gamma_a\bar{\partial}\hat{\theta}) - (z \leftrightarrow \bar{z}), \end{aligned} \tag{5.1}$$

where $\Pi^a_{z/\bar{z}}$ are the supersymmetric momenta. They can be described as the pullback of the bosonic part of the supervielbein

$$\mathbb{E}^A \equiv \mathrm{d}x^M\mathbb{E}_M{}^A \overset{\text{flat}}{=} \big(\overbrace{\mathrm{d}x^a + \mathrm{d}\theta\gamma^a\theta + \mathrm{d}\hat{\theta}\gamma^a\hat{\theta}}^{\Pi^a}\,,\ \mathrm{d}\theta^\alpha\ ,\ \mathrm{d}\hat{\theta}^{\hat{\alpha}}\big) \tag{5.2}$$

to the worldsheet. Letters from the beginning of the alphabet shall denote "flat indices" (with respect to the local frame), while letters from the end of the alphabet will denote "curved indices". This distinction is more relevant for the curved background to be discussed later. As usual, $\gamma^a_{\alpha\beta}$ denotes the off-diagonal chiral block of the 10-dimensional Dirac gamma matrix Γ^a, in a representation where it is real and symmetric (i.e. graded antisymmetric) in the indices α and β.

The Wess-Zumino term $\mathcal{L}_{WZ}$ is responsible for the existence of a local fermionic symmetry, the κ-symmetry. Indeed, the theory contains a number of fermionic constraints $d_{z\alpha}$, $\hat{d}_{\bar{z}\alpha}$. Only half of each set, however, is first class and the constraint algebra is therefore not closed:

$$\{d_{z\alpha}(\sigma), d_{z\beta}(\sigma')\} \quad \propto \quad 2\gamma^a_{\alpha\beta}\Pi_{za}\delta(\sigma - \sigma'). \tag{5.3}$$

Being a spinor in an irreducible representation, $d_{z\alpha}$ cannot covariantly be separated into first and second class and thus does not allow covariant quantization. A long struggle to overcome this problem resulted in the invention of the pure spinor string [27, 30] as an alternative formalism.

Berkovits' pure spinor formalism has two basic ingredients. The first is a free action of the form

$$\begin{aligned} S_{free} &= \int \mathrm{d}^2 z \quad \tfrac{1}{2}\partial x^m\eta_{mn}\bar{\partial}x^n + \bar{\partial}\theta^\alpha p_{z\alpha} + \partial\hat{\theta}^{\hat{\alpha}}\hat{p}_{\bar{z}\hat{\alpha}} \\ &= \int \mathrm{d}^2 z \quad \tfrac{1}{2}\Pi^a_z\eta_{ab}\Pi^b_{\bar{z}} + \mathcal{L}_{WZ} + \bar{\partial}\theta^\alpha d_{z\alpha} + \partial\hat{\theta}^{\hat{\alpha}}\hat{d}_{\bar{z}\hat{\alpha}}, \end{aligned} \tag{5.4}$$

where $p_{z\alpha}$, $\hat{p}_{\bar{z}\alpha}$ are independent variables and $d_{z\alpha} \equiv p_{z\alpha} - (\gamma_a\theta)_\alpha\big(\partial x^a - \frac{1}{2}\theta\gamma^a\partial\theta - \frac{1}{2}\hat{\theta}\gamma^a\partial\hat{\theta}\big)$ and its hatted counterpart have the same algebra as the constraints of the GS-string. In addition, this action

coincides classically with the GS-action for $d_{z\alpha} = \hat{d}_{\bar{z}\hat{\alpha}} = 0$. The second basic ingredient are the BRST operators

$$Q = \oint \mathrm{d}z\ \lambda^\alpha d_{z\alpha}, \quad \hat{Q} = \oint \mathrm{d}\bar{z}\ \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\bar{z}\hat{\alpha}}, \tag{5.5}$$

which implement in some sense the constraints $d_{z\alpha} = \hat{d}_{\bar{z}\hat{\alpha}} = 0$ cohomologically. λ^α and $\hat{\lambda}^{\hat{\alpha}}$ are ghost fields of even parity. Containing also second class constraints, the above BRST operators fail to be nilpotent in general. This can be repaired by constraining the ghost fields to be so-called pure spinors, obeying $\lambda\gamma^a\lambda = \hat{\lambda}\gamma^a\hat{\lambda} = 0$. Like the fermionic coordinates, the ghost fields should be left and right-moving respectively and one thus adds the corresponding ghost term to the free action (5.4): $S_{ps} = S_{\text{free}} + \int d^2z \quad \bar{\partial}\lambda^\alpha\omega_{z\alpha} + \partial\hat{\lambda}^{\hat{\alpha}}\omega_{\bar{z}\hat{\alpha}} + L_{z\bar{z}a}(\lambda\gamma^a\lambda) + \hat{L}_{z\bar{z}a}(\hat{\lambda}\gamma^a\hat{\lambda})$. The implementation of the pure spinor constraints with the help of Lagrange multipliers immediately reveals (by varying with respect to the ghost) a gauge symmetry of the antighosts of the form $\delta_{(\mu)}\omega_{z\alpha} = \mu_{za}(\gamma^a\lambda)_\alpha$ which corresponds to the (first-class) pure-spinor constraints. Because the field equations are basically free, one gets free field operator products after quantization. For the antighost field, this statement is restricted to gauge invariant operators like the ghost current or the Lorentz current. Apart from the central charges, their OPEs look as if there was no pure spinor constraint. To determine the central charges, one has to solve the constraint once (see e.g. [27]).

In order to complete the description for the open string, we still need boundary conditions. For vanishing background a natural choice is to set $\theta = \hat{\theta}$ and $\lambda = \hat{\lambda}$ at the boundary. This can be implemented by the variation of a boundary term that one should add to the action. The precise form of this boundary term is fixed by $N = 1$ supersymmetry, BRST invariance and the antighost gauge symmetry. As the final form of the boundary action is quite lengthy and not very illuminating, we refer to [32] for further details.

The open string in a general background of bulk and boundary fields consists of a bulk part of the same form as a closed string in general background and an additional boundary part. The closed pure spinor superstring in general background was studied first by Berkovits and Howe in [33]. Already at classical level, conservation and nilpotency of the BRST charges implement the type II supergravity constraints. Those, in turn, guarantee 1-loop quantum conformal invariance of the theory [34]. The presentation of the bulk part in the following is based on [35]. The starting point is the most general classically conformally invariant action:

$$\begin{aligned} S_{bulk} \quad = \quad & \int d^2z \quad \frac{1}{2}\partial x^M(\mathbb{G}_{MN}(\vec{x}) + \mathbb{B}_{MN}(\vec{x}))\bar{\partial}x^N + \bar{\partial}x^M \mathbb{E}_M{}^{\boldsymbol{\alpha}}(\vec{x})\, d_{z\boldsymbol{\alpha}} + \partial x^M \mathbb{E}_M{}^{\hat{\boldsymbol{\alpha}}}(\vec{x})\, \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}} + \\ & + d_{z\boldsymbol{\alpha}}\mathbb{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}(\vec{x})\, \hat{d}_{\bar{z}\hat{\boldsymbol{\beta}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\mathbb{C}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}(\vec{x})\, \boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\mathbb{C}}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}(\vec{x})\, \hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}d_{z\boldsymbol{\gamma}} + \\ & + \Big(\bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\beta}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\bar{\partial}x^M\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(\vec{x})\Big) + \Big(\partial\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\partial x^M\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}(\vec{x})\Big)\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} + \\ & + \frac{1}{2}L_{z\bar{z}a}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) + \frac{1}{2}\hat{L}_{\bar{z}z\hat{a}}(\hat{\boldsymbol{\lambda}}\gamma^{\hat{a}}\hat{\boldsymbol{\lambda}}) + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\mathbb{S}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}(\vec{x})\, \boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} \end{aligned} \tag{5.6}$$

The variable $\vec{x}$ contains $x^m, \theta^{\boldsymbol{\mu}}$ and $\hat{\theta}^{\hat{\boldsymbol{\mu}}}$. In addition to the action, we need the two BRST operators. In principle they could contain background fields as well, but it is always possible to reparametrize $d_{z\alpha}$ and $\hat{d}_{\bar{z}\hat{\alpha}}$ such that they have the same form as in the flat case. Consistency of the equations of motion with the pure spinor constraints requires that the background fields $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ are each a sum of a spinorial Lorentz-transformation and dilatation in the last two indices. They can thus be regarded as Lorentz plus scale connections. This property also establishes the antighost gauge symmetry in the general case. BRST invariance of the action requires that the symmetric two-tensor is of the form $\mathbb{G}_{MN} = \mathbb{E}_M{}^a\eta_{ab}\mathbb{E}_N{}^b$. The background fields $\mathbb{E}_M{}^a$, $\mathbb{E}_M{}^{\boldsymbol{\alpha}}$ and $\mathbb{E}_M{}^{\hat{\boldsymbol{\alpha}}}$ can then be combined to a single object $\mathbb{E}_M{}^A$ and regarded as supervielbein. BRST invariance and nilpotency of the BRST transformations put several restrictions on the background fields which turn out to be equivalent to the type II supergravity constraints [33–35].

For the moment, we restrict ourselves to a glance at the propagator. I.e., we are interested in the quadratic part of the action and do not yet need all the constraints. Expanding the coordinates around a constant zero mode, restricting to vanishing zero mode for the fermionic coordinates and the ghosts, choosing a parametrization which corresponds to the WZ-gauge and restricting to the quadratic part, one

arrives at

$$S_{qu}\big|_{\underline{\theta}=\underline{\boldsymbol{\lambda}}=0} = \int d^2z\, \tfrac{1}{2}\partial\zeta^m \left(e_m{}^a(\underline{\vec{x}})\eta_{ab}e_n{}^b(\underline{\vec{x}}) + B_{mn}(\underline{\vec{x}})\right)\bar\partial\zeta^n + d_{z\boldsymbol{\alpha}}P^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}(\underline{\vec{x}})\,\hat d_{\bar z\hat{\boldsymbol{\beta}}} \tag{5.7}$$
$$+\bar\partial\zeta^m\psi_m{}^{\boldsymbol{\alpha}}(\underline{\vec{x}})\,d_{z\boldsymbol{\alpha}} + \bar\partial\zeta^{\boldsymbol{\mu}}\delta_{\boldsymbol{\mu}}{}^{\boldsymbol{\alpha}}d_{z\boldsymbol{\alpha}} + \partial\zeta^m\hat\psi_m{}^{\hat{\boldsymbol{\alpha}}}(\underline{\vec{x}})\,\hat d_{\bar z\hat{\boldsymbol{\alpha}}} + \partial\zeta^{\hat{\boldsymbol{\mu}}}\delta_{\hat{\boldsymbol{\mu}}}{}^{\hat{\boldsymbol{\alpha}}}\hat d_{\bar z\hat{\boldsymbol{\alpha}}} + \bar\partial\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \partial\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}$$

with $x^M(z,\bar z) = \underline{x}^M + \zeta^M(z,\bar z)$. At this stage it becomes visible that the Ramond-Ramond (RR) fields $P^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$ will enter the propagator between the fermionic coordinates. This observation was made for constant RR-fields in [23] for four dimensions (with six compactified on a Calabi-Yau) and in [24] for ten dimensions. The associated anticommutation relations were found to be

$$\{\theta^{\boldsymbol{\alpha}}, \hat\theta^{\hat{\boldsymbol{\beta}}}\} \propto P^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}. \tag{5.8}$$

Turning on the field strength $\mathcal{F}$ modifies the boundary conditions for all world sheet fields and also leads to a RR background dependent shift in the noncommutativity parameter Θ^{mn} [31].

For general backgrounds, one needs to check the consistency of the boundary action with the bulk BRST transformations and the pure spinor constraints. Already for the open pure spinor string in an open string background this is a long story, which was discussed by Berkovits and Pershin in [32]. In addition to the boundary term that was mentioned before they add the integrated open string vertex operator of the form

$$V \propto \int d\tau \quad \dot\theta_+^{\boldsymbol{\alpha}}A_{\boldsymbol{\alpha}}(x,\theta_+) + \Pi_+^m B_m(x,\theta_+) + d_{\boldsymbol{\alpha}}^+W^{\boldsymbol{\alpha}}(x,\theta_+) + \frac{1}{2}(N_+)_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}(\gamma F)_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}}(x,\theta_+) \tag{5.9}$$

to the action. The worldline fields with index '+' are just suitable linear combinations of the left and rightmovers and $(N_+)_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} \propto \boldsymbol{\lambda}_+^{\boldsymbol{\beta}}\boldsymbol{\omega}_{\boldsymbol{\alpha}}^+$. The objects $A_{\boldsymbol{\alpha}}$, B_m, $W^{\boldsymbol{\alpha}}$ and $F^{\boldsymbol{\alpha\beta}}$ are $N=1$ background superfields. The consistency requirements of the boundary action with BRST invariance and the pure spinor constraint leads to the field equations of supersymmetric Born–Infeld for these background superfields.

In order to generalize the result (5.8) to non-constant bulk fields one has to become yet more general, combining the boundary part V with the bulk action (5.6) and studying the consistent boundary conditions and field equations. This is work in progress.

6 Conclusion

In this note we gave an introduction to the Kontsevich product and discussed our proposal for a generalization to the non-associative case. We established cyclicity through second derivative order, which allows for the non-commutative product to be used in the construction of effective actions. We checked that our previous results [10] generalize to non-constant dilaton backgrounds, with the only modification being the prefactor $\exp(-\phi)$ in the measure. We also reviewed the existing results and ideas about generalizations to superstrings, which have been investigated so far for constant background fields.

There is a number of obvious directions for further work. For the bosonic string a non-commutative generalization of the gauge field effective action should be constructed, which presumably is related to derivative corrections to the measure. The non-abelian case should also have interesting implications for commutative non-abelien Born-Infeld actions. A quite demanding task will be the generalization of our results to superstrings in curved RR and B-field backgrounds. On the more mathematical side, it would be interesting to establish cyclicity to all orders in the derivative expansion and if possible explicitly construct the non-associative product.

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